CONTROL OF QUANTUM-MECHANICAL PROCESSES WITHIN EXACTLY-SOLVABLE MODELS WITH MULTI-WELL POTENTIALS

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We review the application of supersymmetric quantum mechanics to various areas of physics, focusing on the advantages of the approach in quantum mechanical systems with multi-well potentials.

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INTRODUCTION

This year we celebrate the centenary of the birth of Academician D.V. Volkov, one of the founders of a new type of symmetry, such as Supersymmetry [1]. Nowadays Supersymmetry [2–4] plays an important role not only in the world of elementary particle physics, but also in nuclear, solid states and condensed matter physics, quantum mechanics, quantum optics and many other areas of contemporary physics. In this brief communication, we aim to point out some features of supersymmetry in the description of various physical systems, which favorably distinguish this approach from the conventional ones.

To recap supersymmetric quantum mechanics (SQM), the approach we promote to describe physical systems, in Section 2 we recall the main points of the construction of isospectral quantum-mechanical Hamiltonians within the standard and extended schemes. In Section 3 we give various examples of systems from different areas of physics in which supersymmetry is hidden. On the contrary, in Section 4 we consider the situation with exact supersymmetry and the outcomes arising from this fact. Conclusions contain a brief discussion of the results and further directions of studies.

A BRIEF RECAP OF SUPERSYMMETRIC QUANTUM MECHANICS

Let's refine main steps in constructing supersymmetric Hamiltonians. We can follow the factorization method proposed by Dirac and Schrodinger in the mid of 20th of the last century. For the sake of simplicity, we will apply the Dirac-Schrodinger formalism to one-dimensional stationary Hamiltonians, when, generally, the Hamiltonian operator is presented by

$$H_0 = -\frac{d^2}{dx^2} + V_0(x). \qquad (1)$$

According to the formalism, one presents the initial Hamiltonian (1) as

$$H_0 = A^\dagger A + \epsilon, \qquad (2)$$

with an analog of the creation/annihilation operators

$$A = \frac{d}{dx} + \beta(x), \quad A^\dagger = -\frac{d}{dx} + \beta(x). \qquad (3)$$

The unknown function $\beta(x)$, entering eq. (3), can be found from the Riccati equation

$$-\frac{d}{dx}\beta(x) + \beta^2(x) = V_0(x) - \epsilon. \qquad (4)$$

Though this construction is universal, and can be applied to any quantum-mechanical Hamiltonian, the general solution to the Riccati equation is absent in the analytic form. Put it differently, one cannot resolve eq. (4) for an arbitrary potential $V_0(x)$, as well as the Schrodinger equation with Hamiltonian (1).

However, there is a set of quantum-mechanical potentials, for which the exact solutions to the Riccati equation exist. These potentials are referred to as exactly-solvable potentials of Quantum Mechanics (QM). And in this case, we can apply the factorization scheme and determine the operators (3) and the factorization energy ϵ explicitly [5].

But it is not the end of the story. In 1984 Mielnik [6] proposed the way of constructing a new Hamiltonian from the original one, if a particular solution to the Riccati equation has known. If we denote this particular solution as $\beta_0(x)$, then the new Hamiltonian will receive the structure of

$$H_1 = AA^\dagger + \epsilon = -\frac{d^2}{dx^2} + V_1(x) \qquad (5)$$

with the new potential

$$V_1(x) = V_0(x) + 2\frac{d}{dx}\beta_0(x). \qquad (6)$$

The so constructed new potential can be absolutely different in shape. However, the spectra of Hamiltonians H_0 and H_1 turn out to be related to each other via the so-called intertwining relations:

$$H_1 A = A H_0, \;\; H_0 A^\dagger = A^\dagger H_1. \qquad (7)$$

When the starting Hamiltonian H_0 is an exactly-solvable one, we can choose

$$\beta_0 = -\frac{\varphi'_\epsilon(x)}{\varphi_\epsilon(x)}, \qquad (8)$$

and the still arbitrary factorization energy ϵ becomes a part of the spectrum of one of the Hamiltonian partners. Specifically, as $\epsilon < E_0$ (where E_0 is the ground state energy of H_0), the factorization energy is the ground state energy of the new Hamiltonian H_1. As a result, the intertwined Hamiltonians H_0 and H_1 are (almost) isospectral; their spectra differ only in the ground energy state Fig. 1).

So far, we did not even mention Supersymmetry. However, there is a very close connection [7, 8] between operators A and $A^\dagger$ of (3) and the so-called *supercharges* Q and $Q^\dagger$ forming the part of the minimal Supersymmetry algebra:

$$\{Q, Q^\dagger\} = H. \qquad (9)$$

Then, the paired from the point of view of relations (7) Hamiltonians form the so-called *supermultiplet*. Since properties of a (super)multiplet members have to be the same, it is not surprising that the spectra of the Hamiltonians-superpartners are (almost) the same.

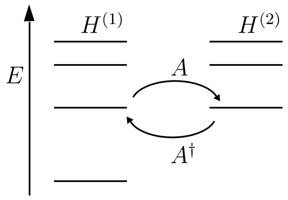

Fig. 1. Schematical spectrum of two intertwined Hamiltonians related by eqs. (7). Here $H^{(1)} \equiv H_1$, $H^{(2)} \equiv H_0$

One may wonder, does the presence of Supersymmetry turn out to be so important? We know that experimentally Supersymmetry is still a hidden symmetry of the Nature. However, the fact of absence of superpartners in the realm of particle physics tells us, first of all, about the broken Supersymmetry on the LHS action scale. So that, Supersymmetry (SUSY) may be non-manifest, and this fact can be illustrated by several examples.

SYSTEMS WITH HIDDEN SUSY

First example of hidden isospectrality is borrowed from gravitational physics. It is well known that gravitational waves are spin-2 fluctuations over a gravitational background. If, for simplicity, the non-trivial background is chosen to be that of a Schwarzschild black hole, it is determined by the so-called red-shift factor $f(r)$. In the linear approximation, dynamics of spin-2 fluctuations h_s over the Schwarzschild gravitational background is determined by a Schrodinger-type equation [9, 10]

$$\left[\frac{\partial^2}{\partial r_*^2} + \omega^2 - V_s(r)\right] h_s = 0$$

with the Wheeler coordinate $r_* \in (-\infty, +\infty)$; $s = \pm 2$. The effective potential of the axial perturbations over the background metric (linearly polarized gravitational waves) comes as follows:

$$V_{+2}(r) = -\frac{3f(r)\partial_r f(r)}{r} + l(l+1)\frac{f(r)}{r^2}. \quad (10)$$

For the circularly polarized gravitational waves the effective potential becomes

$$V_{-2}(r) = \frac{2f(r)}{r^3}\frac{9M^3 + 3c^2Mr^2 + c^2(1+c)r^3 + 9M^2cr}{(3M+cr)^2},$$

$$c = \frac{l(l+1)}{2} - 1. \quad (11)$$

Apparently, in eqs. (10), (11) l are integers, starting from $l = 2$.

If we insert into eqs. (10) and (11) the red-shift factor for the Schwarzschild black hole, $f(r) = 1 - r_+/r$, we encounter the difference between the effective potentials from (10) and (11). However, the effective Hamiltonians are (almost) isospectral, that can be find from the analysis of the effective potential shapes (Fig. 2). There are a lot of debates on the origin of such an isospectrality. It is shown for simple backgrounds [12]; for more exotic configuration of gravitational field the approach fails [13].

Another example of dealing with physical systems with hidden SUSY comes from Quantum Optics. Quantum Optics by itself describes the natural interaction of bosonic (photons) and fermionic (electrons) subsystems of a medium. In the space of parameters of the light-matter interaction it may arise the Bose/Fermi Duality, that transforms, under some specific conditions, into SUSY [14].

Following [14], let's consider the generalized Rabi model, which is determined by the following Hamiltonian describing a 2-level system interacting with a monochromatic wave:

$$H = \hbar\omega a^\dagger a + \frac{\Delta}{2}\sigma_z + g_1(a^\dagger\sigma_- + a\sigma_+) + g_2(a^\dagger\sigma_+ + a\sigma_-). \quad (12)$$

In (12), Δ is the levels gap; ω is the boson field frequency; $a^\dagger$ and a are the bosonic ladder operators; and, finally, $g_{1,2}$ are arbitrary constants of the light-matter interaction.

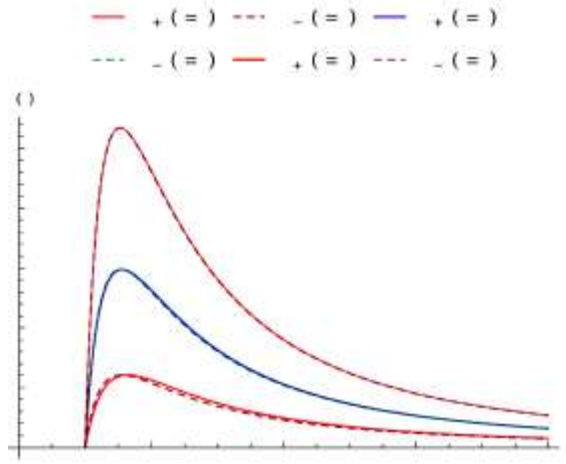

Fig. 2. The shapes of effective potentials for axial and polar spin-2 perturbations over the Schwarzschild background. Borrowed from Ref. [11]

In framework of Quantum Optics, the generalized model with Hamiltonian (12) describes the dipole interaction of a monochromatic wave with the bi-level emitter. In the limit of zero-valued constants of interaction, the considered generalized model turns into the Jaynes-Cummings model. When two constants are the same, we get the Rabi model. If one of the interaction constants is small (say, $g_2 \to 0$), the near-resonant consideration (at $\omega \sim \Delta$) corresponds to the rotating-wave approximation (RWA). In the strong interaction constants regime both interaction terms (co- and contra-rotating) should be taken into account.

As it has been proved in [14], SUSY is a symmetry of the generalized Rabi model under the following condition:

$$g_1^2 - g_2^2 = \Delta \cdot \omega.$$

Then, the supercharge has the form of 4×4 matrix

$$Q = \begin{pmatrix} 0 & \hat{q} \\ 0 & 0 \end{pmatrix}, \quad \hat{q} = \begin{pmatrix} g_1/\sqrt{\omega} & \sqrt{\omega}\, a \\ \sqrt{\omega}\, a & g_2/\sqrt{\omega} \end{pmatrix}.$$

As usual for SUSY (cf. eq. (9)),

$$\{Q, Q^\dagger\} = H.$$

For $g_1 = g_2$, SUSY is realized with $\Delta = 0$. And Hamiltonian (12) turns into the standard Harmonic Oscillator with the energy shift.

The next example comes from Condensed Matter Physics. It turns out that SUSY is a hidden symmetry in a topological insulator with Josephson junctions [15]. Such a system can be described by the Bogoliubov-de

Gennes equations, which, after the appropriate simplifications (see [15] for details), can be reduced to the system of differential equations

$$(E + iv\partial_x)f(x) + \Delta(x)\varphi(x) = 0,$$
$$(E - iv\partial_x)\varphi(x) + \Delta^*(x)f(x) = 0.$$

Here E is the energy of moving with the velocity v in the positive x direction spinning mode; $\Delta(x)$ is the complex energy gap. It is easy to transform this system of equations in the equation of Witten's Supersymmetric Quantum Mechanics (SQM) [7]

$$E^2\psi(x) = \left(-v^2\partial_x^2 + \widehat{W}^2 + iv\sigma_z\frac{\partial\widehat{W}}{\partial x}\right)\psi(x),$$

where $\psi(x)$ is the spinning mode state

$$\psi(x) = \begin{pmatrix} f(x) \\ \varphi(x) \end{pmatrix},$$

and the *superpotential* $\widehat{W}(x)$ is given by

$$\widehat{W}(x) = \begin{pmatrix} 0 & \Delta(x) \\ \Delta^*(x) & 0 \end{pmatrix}.$$

This series of examples could be continued by notable systems with hidden SUSY in Quantum Mechanics [16], nuclear physics [17] and mesoscopy [18].

EXACTLY-SOLVABLE MODELS OF SQM WITH MULTI-WELL POTENTIALS

In this part of the review, we will focus on models of N=4 Supersymmetric Quantum Mechanics [19], which are based on the Harmonic Oscillator potential $V_0(x) = x^2$ and its deformations.

First of all, let's describe the technique behind the deformation of the potential shape. Looking at Fig. 1, one may notice that the original and the paired Hamiltonians are different in spectra by just one level. And if this new additional level has the energy less than the ground state energy of the original Hamiltonian, it defines the vacuum state of the paired Hamiltonian with new potential (6). To keep the coincident part of the both Hamiltonians spectra, the shape of the new potential shall be changed. The level of deformation depends on the interplay between the unfixed parameters that naturally arise in this scheme upon the definition of new, non-renormalizable, wave function $\varphi_\epsilon(x)$, corresponding the new ground state with the factorization energy ϵ. For instance, the Harmonic Oscillator potential describes the system with one potential well; its deformation may potentially form another well. So that, by controlled adding an additional level, we are able to get the (almost) isospectral quantum mechanical system with two-well potential. However, the clear impact of Quantum Mechanics in this case will consists in possible *tunneling effects* between the wells of the paired to the Harmonic Oscillator potential. Further, by adding a new level, lower than the ground level of the Hamiltonian H_1, one can form another, paired to H_1, Hamiltonian H_2 with another potential, the shape of which will be different from the potential V_1. Here, the interplay between parameters of the solutions may result as in two-well as well as in three-well potentials of different shapes. In the latter case, one is able to study more complex tunneling effects that provides a lot of possibilities to control quantum-mechanical processes.

As an example of such controlling let us present results of modelling a wave package behavior in a symmetric and a non-symmetric two-well potentials for isospectal Hamiltonians, Figs. 3 and 4. Due to the difference in probabilities of the first, second and third levels in different wells of symmetric and asymmetric potentials, the probability flow of wave packages prepared with the corresponding energies essentially varies. It models the behavior of a *quantum diode*.

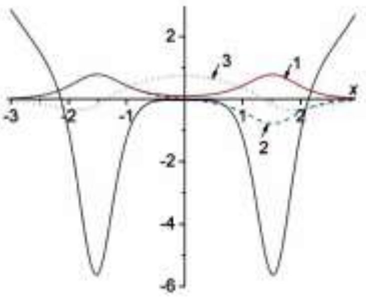

Fig. 3. The relative probability of the first three levels in different wells of a symmetric two-well potential. See Ref. [20] for details

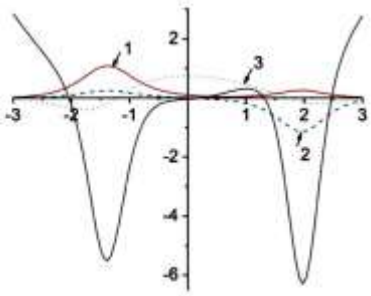

Fig. 4. The relative probability of the first three levels in different wells of an asymmetric two-well potential. See Ref. [20] for details

When the number of wells in the isospectral Hamiltonian potential becomes equal to three, the situation becomes more complicated. Here, with tunneling process, one can model properties of a *quantum transistor*, with different values of "current" flow in different wells, see Fig. 5.

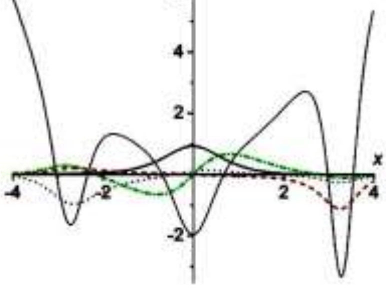

Fig. 5. The relative probability of the first three levels in different wells of an asymmetric three-well potential. See Ref. [20] for details

One can generalized the description with adding the temporal dependence. For instance, it could be an external periodic driving force that, in real devices, is reproduced by a laser EM field. The interplay of parameters in the extended by the external field frequency set allows one to reach the phenomenon of the so-called Coherent Tunneling Destruction (CTD) [21], when without changing the quantum character of the system it makes it possible to localize the initial wave packet in one of the wells. As an illustration of the CTD, in Fig. 6 we present the data of numerical simulations borrowed from Ref. [20].

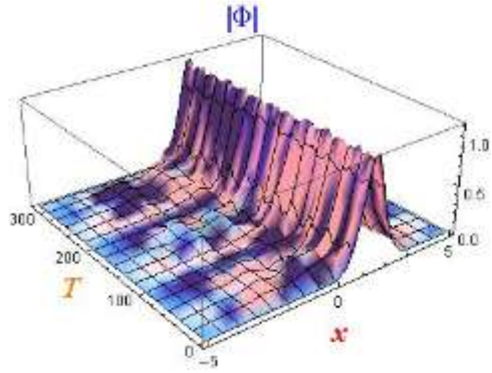

Fig. 6. Numerical simulation of evolution of the Gaussian wave packet $\Phi(x,T)$ in the external periodic driving force. We refer the reader to [20] for details

Hence, the formalism of Supersymmetric Quantum Mechanics turns out to be helpful in modelling different processes such as tunneling, particles flow, diffusion and so on, in the controllable manner. The latter is achieved by the exactly-solvable character of the models in hands, when the Green's functions are constructed out the explicitly known analytic expressions for the wave functions. Thus, evolution in such systems becomes, if not deterministic, then very predictable.

CONCLUSIONS

In conclusion, let us recall the main advantages of Supersymmetry that make this approach preferable to others.

First, as it has been pointed out in one of the underlying work on Supersymmetry [22], supersymmetric models are superrenormalizable ones: the UV divergency problem, actual for standard field theories including the Standard Model, is completely absent for them. Unfortunately, our Universe is not supersymmetric; but this fact is about the energy scale on which SUSY is broken.

Second, application of Supersymmetry as a tool to investigate various physical models in different regimes and on different energy scales has shown its self-consistency and efficiency. We can just cite a few quotes from the modern literature in favor of this claim. For instance, the authors of Ref. [14] write: " … optical and condensed matter systems at the SUSY points can be used for quantum information technology and can open an avenue for quantum simulation of the SUSY field theories." In Ref. [23] one can read off: " … the atomic nucleus ^{195}Pt represents an excellent example of the dynamical U(6/12) supersymmetry. … certainly the best documented example of the manifestation of dynamical supersymmetry in atomic nuclei." As it comes from reading Ref. [24], the potential of supersymmetry has not yet been definitively revealed, since the author "… brings the attention to the role of supersymmetry in quantum computation and quantum information more broadly, a subject much underexplored."

And finally, our research on the question posed in the title of these notes, shows the perspectives of applying the SQM formalism in studying quantum-mechanical problems. Recently, two of us have extended the approach to include the temporal dependence into the game [25], that made it possible to relate CPT-invariant stationary Hamiltonians to their PT-invariant non-stationary partners. It opens new avenues in investigations of quantum-mechanical models with complex-valued potentials, having more reach structure of physical phenomena, and being applicable to an essentially wide class of physical systems.

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REFERENCES

1. С.И. Волкова, А.Ю. Нурмагамбетов. *Дмитрий Васильевич Волков. Статьи, очерки, воспоминания*. Харьков: Издательство Тимченко А.Н., 2007, 706 с.
2. Ю.А. Гольфанд, Е.П. Лихтман. Расширение алгебры генераторов Пуанкаре и нарушение P-инвариантности // *Письма в ЖЭТФ*. 1971, т. 13, №8, с. 452-455.
3. Д.В. Волков, В.П. Акулов. О возможном универсальном взаимодействии нейтрино // *Письма в ЖЭТФ*. 1972, т. 16, №11, с. 621-624.
4. J. Wess, B. Zumino. Supergauge transformations in four dimensions // *Nucl. Phys.* 1974, v. 70, p. 39-49.
5. L. Infeld, T.E. Hull. The factorization method // *Rev. Mod. Phys.* 1951, v. 23, p. 21-68.
6. B. Mielnik. Factorization method and new potentials with the oscillator spectrum // *J. Math. Phys.* 1984, v. 25, p. 3387-3389.
7. E. Witten. Dynamical Breaking of Supersymmetry // *Nucl. Phys.* 1981, v. 188, p. 513-554.
8. Л.Э. Генденштейн, И.В. Криве. Суперсимметрия в квантовой механике // *УФН*. 1985, т. 146, №4, с. 553-590.
9. T. Regge, J.A. Wheeler. Stability of a Schwarzschild singularity // *Phys. Rev.* 1957, v. 108, p. 1063-1069.
10. F.J. Zerilli. Effective potential for even-parity Regge-Wheeler gravitational perturbation equations // *Phys. Rev. Lett.* 1970, v. 24, p. 737-738.
11. A.M. Arslanaliev, A.J. Nurmagambetov. Scattering on Quasi-Spherical Black-Holes: Features and Beyond // *MDPI Physics*. 2021, v. 3, p. 17-41.
12. K. Glampedakis, A.D. Johnson, D. Kennefick. Darboux transformation in black hole perturbation theory // *Phys. Rev. D*. 2017, v. 96, p. 024036.
13. D. Li, A. Hussain, P. Wagle, Y. Chen, N. Yunes, A. Zimmerman. *Isospectrality breaking in the Teukolsky formalism:* Preprint arXiv:2310.06033 [gr-qc], 29 pages.

14. M. Tomka, M. Pletyukhov, V. Gritsev. Supersymmetry in quantum optics and in spin-orbit coupled systems // *Scientific Reports*. 2015, v. 5, p. 13097.
15. A.V. Galaktionov. Supersymmetric Hamiltonian solutions simulated by Andreev bound states // *Phys. Rev. B*. 2020, v. 101, p. 134501.
16. E. D'Hoker, L. Vinet. Supersymmetry of the Pauli Equation in the Presence of a Magnetic Monopole// *Phys. Lett. B*. 1984, v. 137, p. 72-76.
17. F. Iachello. Dynamical Supersymmetries in Nuclei // *Phys. Rev. Lett*. 1980, v. 44, p. 772-775.
18. А.Ф. Андреев, М.Ю. Каган. Супергидродинамика ^{3}He-A // *ЖЭТФ*. 1987, т. 93, с. 895-903.
19. A.I. Pashnev. One-dimensional Supersymmetric Quantum Mechanics With N≥2 // *Theor. Math. Phys.* 1986, v. 69, p. 1172-1175.
20. V.P. Berezovoj, M.I. Konchatnij, A.J. Nurmagambetov. Hallmarks of tunneling dynamics with broken reflective symmetry // *Nucl. Phys. B*. 2021, v. 969, p. 115483.
21. F. Grossmann, T. Dittrich, P. Jung. P. Hanggi. Coherent Destruction of Tunneling // *Phys. Rev. Lett*. 1991, v. 67, p. 516-519.
22. J. Wess, B. Zumino. A lagrangian model invariant under supergauge transformations // *Phys. Lett. B.* 1974, v. 49, p. 52-54.
23. J. Jolie. Experimental evidence for dynamical supersymmetries as illustrated by ^{195}Pt // *AIP Conf. Proc.* 2019, v. 2150, p. 020012.
24. P.M. Crichigno. *Supersymmetry and Quantum Computation*. Preprint arXiv:2011.01239 [quant-ph], 8 pages.
25. V.P. Berezovoj, A.J. Nurmagambetov. Nonstationary SQM/IST Correspondence and CPT/PT -Invariant Paired Hamiltonians on the Line // *PTEP*. 2024, v. 6, p. 063A02.