NUMERICAL ALGORITHM FOR PLASMA COOLING AND DECAY IN PLASMA LENS

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The numerical algorithm for investigation of plasma cooling and decay after capillary discharge is described. The axially symmetric hydrodynamic model is considered. The set of hydrodynamic equations in partial derivatives is discretized and changed by set of ordinary temporal differential equations that is solved numerically. The analytical fitting to calculate spatial derivatives is used. The main spatial regions that are problem for numerical simulation are pointed out and analyzed. The way to avoid numerical problem near the capillary axis center and capillary wall boundary is presented and analyzed.

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INTRODUCTION

The new plasma-based acceleration methods, either laser or beam driven, create new possibilities comparing to a traditional radio-frequency one. It allows to build compact accelerators. The strong electric field with strength about GV/cm allows to produce stable, ultrashort, high quality GeV electron beams with very low emittance and high brightness using high acceleration rate [1-4]. The laser pulse or accelerated electron beam are focused into narrow capillary channel and create plasma in them. The capillary discharge is used as a waveguide for short laser pulses. On another hand, a capillary discharge may be used as a plasma lens to focus and transport accelerated beam [1-3].

The capillary discharge is produced in the thin cylindrical tube, that has diameter about 1 mm and length of several centimeters, by external electric field. The capillary is filled by hydrogen or argon gas. This field is generated by high voltage pulse, that is supplied to external plane or cone electrodes that is placed on the cylindrical capillary ends, and has duration about one microsecond. External voltage may vary from several hundred volts up to about 10 kV. Excited current may start from several hundred Amperes and achieve several kilo Amperes. During several hundred nanoseconds the ionization of neutral gas and its heating up to several electron-Volts takes place. Plasma in this case may be almost fully ionized.

In this talk the plasma decay after capillary discharge is investigated to define the recovery of acceleration conditions.

1. THE PROBLEM

To investigate the plasma cooling and decay the axially symmetric model is used. The capillary is assumed to be infinite in longitudinal direction. The working gas is hydrogen. The plasma temperature changing along capillary radius is from 3…5 eV at capillary center up to 0.1…0.2 eV on it wall. In this case it is supposed that plasma contains only atomic hydrogen as neutral particle. Besides neutral atomic hydrogen the plasma contains hydrogen ions and electrons. To investigate the plasma decay the three-body recombination and thermal conductivity processes are considered. Plasma is considered as quasi-neutral and isothermal. The initial pressure of cold gas is about 1 kPa.

To describe the mentioned phenomena the one-dimensional hydrodynamic equations in cylindrical coordinate system is used and may be presented as following

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \quad (1a)$$

$$\frac{\partial n_e}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r n_e v) = -\beta_{tr} n_e^3, \quad (1b)$$

$$\frac{\partial n_n}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r n_n v) = \beta_{tr} n_e^3, \quad (1c)$$

$$\frac{3}{2}k_b(2n_e+n_n)\frac{\partial T}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{5}{2}pv\right]-v\frac{\partial p}{\partial r}= 3k_bT\frac{1}{r}\frac{\partial}{\partial r}(rn_ev)+\frac{3}{2}k_bT\frac{1}{r}\frac{\partial}{\partial r}(rn_nv)+ \left(\frac{3}{2}k_bT+I_H\right)\beta_{tr}n_e^3+\frac{1}{r}\frac{\partial}{\partial r}\left(r\kappa\frac{\partial T}{\partial r}\right). \quad (1d)$$

Here r is radius, t is time, n_e, n_n are charged (electron and ion) and neutral particles densities respectively, v is plasma velocity, ρ is mass plasma density, p is pressure, β_{tr} is three-body recombination rate, k_B is Boltzmann constant, κ is plasma thermal conductivity, T is plasma temperature. The boundary condition for these equations are: the velocity on capillary bound is equals to zero, the temperature is kept constant 0.1…0.2 eV. The initial pressure distribution is assumed uniform and equals to 144000 Pa. The temperature, ionization degree (respectively charged particle density) are set analytically close to experimental measurements. The spatial profile for neutral particles calculated from the temperature and plasma densities. The initial distribution velocity was assumed to be zero. The radial derivatives for all hydrodynamics variables on the capillary axis center are kept to zero (due to axial symmetry). It will be discussed in details below.

2. SOME ALGORITHM FEATURES

For numerical solution of the set of equation (1), the spatial discretization along capillary radius was carried out. The initial set of equations in partial derivatives is changed by the set of ordinary differential equations for every hydrodynamic variable at every spatial point. This set of equations is solved numerically. The spatial partial derivatives contained in the discretized set of equation are defined numerically on every time step.

To calculate the values for all spatial derivatives the fitting by analytical formulae is used. This way was chosen instead finite differences to prevent numerical instability near the capillary axis. Numerical simulation showed that there are two problem regions along the capillary radius. One of them is near the capillary axis center and second one is located close to the capillary wall.

The difficulties near axis center are evident. On other hand the numerical problems in the region close to capillary wall are sufficiently complex, they were detected after numerous numerical studying. First of all, let us consider numerical problem in spatial region close to the capillary axis center. The reason for these problems is the following terms in hydrodynamics equations:

$$\frac{1}{r}\frac{\partial}{\partial r}(rnv),\frac{1}{r}\frac{\partial}{\partial r}[rpv],\frac{1}{r}\frac{\partial}{\partial r}\left(r\kappa\frac{\partial T}{\partial r}\right). \quad (2)$$

They contain small value in denominators. If spatial derivatives are calculated with low precision it may cause numerical instability in this region. To avoid these difficulties the fitting analytical formula is used. The radial profile for every hydrodynamic variable is presented as power series relatively capillary radius and has the following form:

$$AnyVariable(r,t) = \sum_{j=0}^{N} c_j\,(t)r^j. \quad (3)$$

Here $c_j(t)$ are fitting coefficients, depending on time and are calculated on every time step. To define the coefficients the list square method is applied, that is expressed by following minimization problem for fitting coefficients:

$$F(c_1,c_2,..,c_N)=\sum_{i=1}^{M}\left(\sum_{j=0}^{N} c_j r_i^j - Value_i\right)^2 + \lambda\sum_j^N c_j^2 \to min. \quad (4)$$

To minimize function $F(c_1, c_2,\ldots, c_N)$ additional constraints can be added, scaling of input data along the horizontal and vertical axis may be used.

Here N is coefficient number, M is spatial point number, $Value_i$ is value of any hydrodynamic variable in the point r_i, λ is regularization parameter.

The first term in Eq.(4) immediately is responsible for the list square method, the second term corresponds Tikhonov's regularization method that may be used if coefficients fitting search is unstable and minimization problem is ill posed [5, 6]. The function to be minimized may be supplemented by additional constraints, that may express spatial analytical properties of hydrodynamic variables. Besides due to linearity fitting problem the scaling along both axes may be used.

For numerical simulation the MATHEMATICA version 13.0 package is used. The build-in function “findFitt” allows to realize above mentioned features. The numerous numerical investigations allow to reach physically reasonable results in both problem regions. However, it is necessary to extend these properties to long times about a few microseconds, that correspond to experimental results.

3. PRELIMINARY NUMERICAL RESULTS

First of all, let us briefly consider the preliminary numerical investigation for much process time. The experiment duration is a few microseconds. In Figs. 1-3 temperature, electron density and pressure profiles the simulation for a time moment t = 2 μs are presented. As it is seen the start spatial point is shifted from capillary axis center. This displacement is approximately equal 5% from capillary radius. This displacement allows to reduce incorrect numerical results near axis. Nevertheless, problems are remaining in this region. Especially it is seen on Figs.1 and 2, The 5% displacement was chosen due to more or less physically correct results obtained in this region.

The biggest problem is observed in Fig. 3 where the radial pressure profile is presented. Regardless of more or less reasonable dynamics of temperature and electron density, when these values are decreased during plasma decay process, the significant drawback is observed in Fig. 3. The pressure distribution is remaining almost uniform and average pressure level is decreased at this time from the initial value of about 144000 Pa. But on another side, there is large pressure jump on the capillary wall. At short times this jump may be neglected. This jump is appearing always from initial program cycles. It does not depend from the spatial point distribution, calculation time step and another program parameters. To avoid this jump, the additional analytical and numerical investigations were required that will not be presented here. So, main way how to obtain physically correct results will be considered latter.

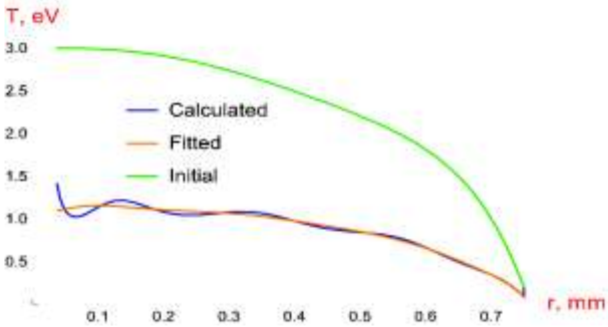

Fig. 1. Temperature profile at t = 2 μs. Green line corresponds to initial temperature distribution, blue line is calculated one, brown line is obtained by fitting

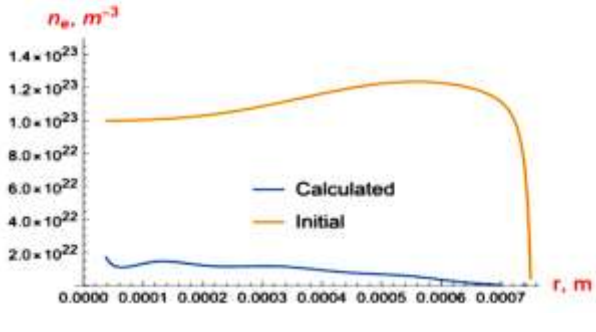

Fig. 2. Electron density profile at t =2 μs. Blue line corresponds to calculated result, brown line corresponds to the initial electron density profile

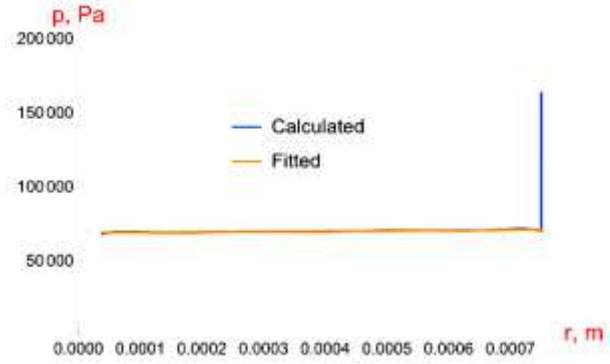

Fig. 3. Pressure profile at t = 2 μs. Blue line corresponds to calculated result, brown line is fitted result

4. PROBLEM WITH SOLUTION IN THE REGION CLOSE TO THE CAPILLARY WALL BOUNDARY

As it follows from the boundary condition for the velocity and momentum equation (1a) the pressure gradient on the capillary wall should be kept to zero. This circumstance was checked in the program. The detailed numerical investigations showed that this condition is violated during program running. Moreover, the pressure gradient is continuously growing. This fact is not understood from numerical algorithm point of view. To prevent this growth the zero gradient condition may be used as additional constraint (equality) for fitting coefficients that has following form

$$\frac{\partial P}{\partial r}(r = r_b) = \sum_{i=1}^{n} i c_i(t)\, r_b^{i-1},\ c_1 = 0, \qquad (5)$$

where r_b is the radius of the capillary boundary. In Eq. (5) another constraint is presented that takes into account equality to zero of the pressure gradient on the capillary axis center. But as it was turning out only this one condition is not sufficient to obtain proper result. For convenience, the pressure may be presented in the following form (pressure deviation from the average level):

$$\Delta p = p - \overline{p},\ \ \overline{p} = \frac{\sum_{j=1}^{M} p_j}{M}, \qquad (6)$$

where $\overline{p}$ is the average pressure level. The result of pressure deviation fitting with constraints (5) are presented in Fig. 4.

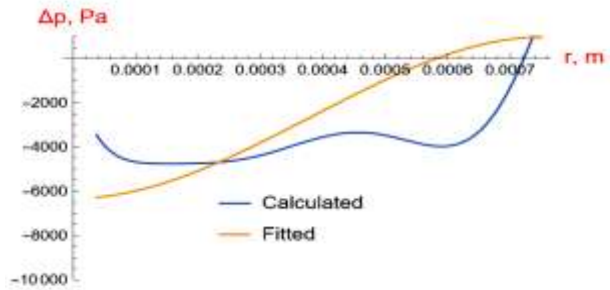

Fig.4. One example of pressure profile fitting for pressure deviation from the average pressure level. Blue line corresponds to the calculated result, brown line is the fitted one

As it is seen in this figure the conditions of Eq. (5) are satisfied. But fitting results is not sufficiently satisfactory (it is necessary to remind that the average level is about 144000 Pa). Using Tikhonov's regularization and scaling along the radial and pressure deviation axes allowed significantly to improve fitting process. The corresponding result is shown in Fig. 5.

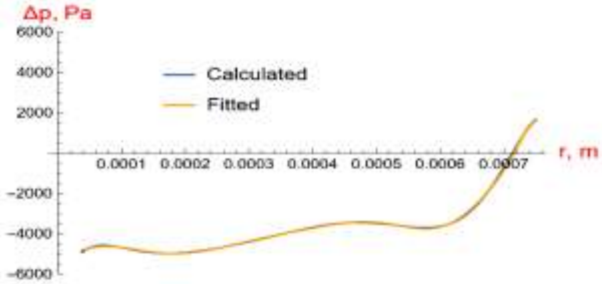

Fig. 5. The best fitting results for the pressure profile. Lines for calculated (blue line) and fitted (brown line) pressure deviation coincide

Regularization parameter and scaling coefficients were picked up manually. The gradient of profile presented in Fig. 5 are shown in Fig. 6.

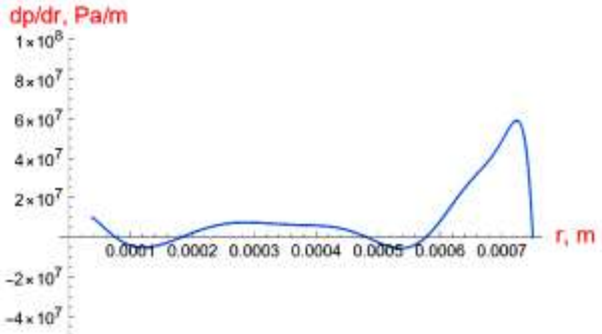

Fig. 6. Pressure gradient for the profile presented in Fig.5

5. TIME EXTENDED INVESTIGATIONS WITH MENTIONED CONSTRAINTS

Numerous numerical investigations showed stable operation of suggested approach in the wide range of pressures near the capillary wall boundary. But the difficulties remain near the capillary axis center.

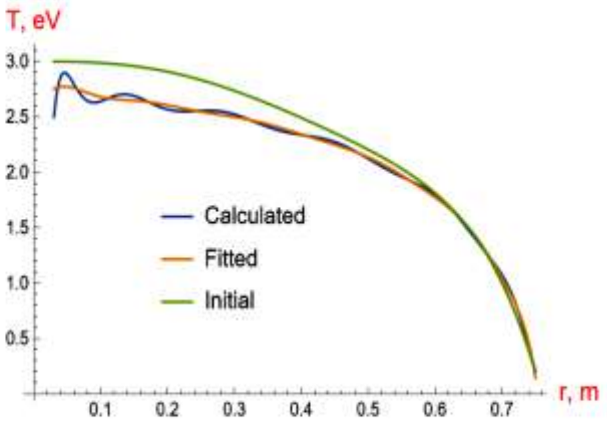

Fig. 7. Temperature profile for t ≈ 60 ns. Green line corresponds to the initial temperature profile, blue line is the calculated result, and brown line is the fitting

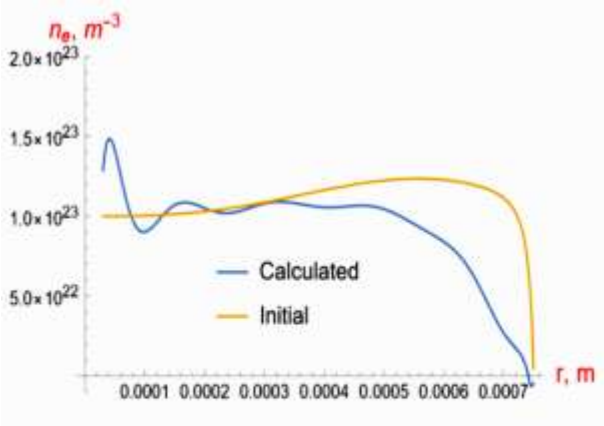

Fig. 8. Electron density profile for t ≈ 60 ns. Blue line corresponds to calculated result and brown line is the initial electron density profile

In Figs. 7, 8 the temperature and electron density profiles are shown at t≈60 ns, respectively. As it is seen in the spatial region close to the axis center the result is physically incorrect.

To solve this problem the presented above approach is used. Let us return to terms in Eq. (2). As it was mentioned in the section 3 these terms are reason of the numerical problems near the capillary axis center due to the possible mistakes in variables in nominators of these terms. So, the higher precision is required to fit the following variables

$$rnv, rpv, r\kappa\frac{\partial T}{\partial r}. \quad (7)$$

It is possible analytically to show that in the axially symmetric case the following limits on the axis center for terms in Eq. (2) are valid

$$\frac{1}{r}\frac{\partial}{\partial r}(rnv)|_{r=0}=0, \frac{1}{r}\frac{\partial}{\partial r}(rpv)|_{r=0}=0,$$
$$\lim_{r\to 0}\frac{1}{r}\frac{\partial}{\partial r}\left(r\kappa\frac{\partial T}{\partial r}\right)=2\kappa(r=0)\frac{\partial^2 T}{\partial r^2}(r=0). \quad (8)$$

The analytical asymptotic for different hydrodynamic variables are presented in the following expressions:

$$v(r=0)=0, \frac{\partial v}{\partial r}(r=0)=0, \ v\sim r^2, \quad (9)$$
$$p(r=0)=p_0, \frac{\partial p}{\partial r}(r=0)=0, p\sim p_0+\text{Const } r^2, \quad (10)$$
$$n_{e,i}(r=0)=n_{e,i0}, rn_{e,n}v(r=0)=0, rn_{e,n}v\sim r^3, \quad (11)$$
$$rpv(r=0)=0, \ rpv\sim r^3, \quad (12)$$
$$T(r=0)=T_0, \frac{\partial T}{\partial r}(r=0)=0, T\sim T_0+\text{Const } r^2, \quad (13)$$
$$r\kappa\frac{\partial T}{\partial r}\sim r^2. \quad (14)$$

Here p_0 is the pressure, T_0 is the temperature, $n_{e,i0}$ is electron and neutral particles density on the axis center. From the expressions (9) – (14) the constraints for fitting coefficients for listed hydrodynamics variables are following:

for velocity v → cv_0, $cv_1=0$,
for pressure p → $cp_1=0$,
for fluxes $rn_{e,n}v$ → $cn_{e,n}v_0, cn_{e,n}v_1, cn_{e,n}v_2=0$,
for rpv → $crpv_0, crpv_1, crpv_2=0$,
for thermal flux $r\kappa\frac{\partial T}{\partial r}$ → cth_0, $cth_1=0$.

Expressions presented on the right hand are constraints for the hydrodynamic variables taking into account its analytical behavior in the region close to the capillary axis center.

In Figs. 9, 10 the pressure deviation and velocity profiles demonstrate influence of condition near the capillary axis center on the numerical results. Additional constraints in the region close to the axis center allowed to bring spatial start point immediately close to capillary axis center on distance about 0.1% of the capillary radius. This start point position is almost not distinguishable from the position of the axis center.

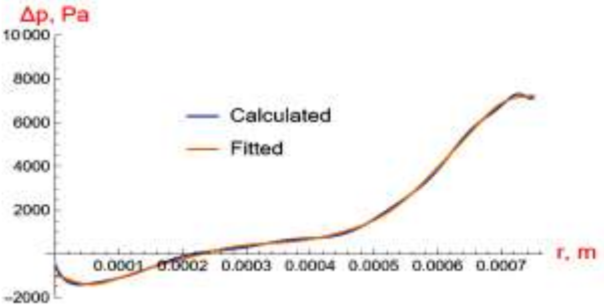

Fig. 9. Example of pressure deviation profile fitting near capillary axis for t = 9.3 ns. Blue line corresponds the numerical results, brown line is the fitting

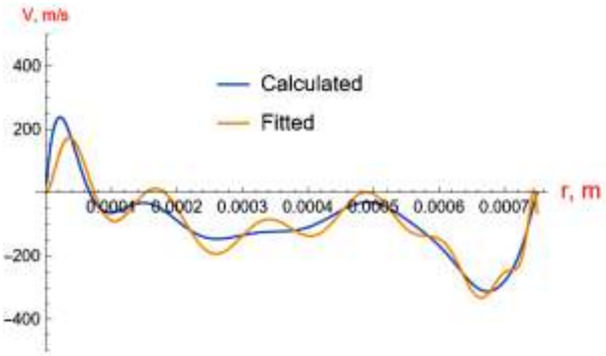

Fig. 10. Example of velocity profile fitting near capillary axis center for t = 9.3 ns. Blue line corresponds the numerical results, brown line is the fitting

Despite reached improvement, some problems remain. The velocity profile near the axis center should be similar to parabola. Actually, for the fitting profile it is true. Both linear terms in power series are zero. The spatial region where quadratic term dominates is very small, thus contribution of higher power terms is significant in the immediate vicinity from the axis center. On the other hand, the precision of the velocity fitting is not sufficient. The possible reason may be the complex velocity profile for fitting.

CONCLUSIONS

The algorithm for numerical investigation of plasma cooling and decay after the capillary discharge in plasma lens is considered. The axially symmetric hydrodynamic model is used. The initial set of equations in partial derivatives are discretized along the capillary radius and changed by the temporal set of ordinary differential equations. The analytical fitting of spatial profiles for hydrodynamic variables is used. These analytical formulae are used to obtain spatial derivatives.

The detected numerical problems are presented. It is pointed out, that there are two spatial regions where numerical problems are arising. One of these regions is located near the capillary axis center, another near capillary wall boundary. The reason of numerical difficulties is investigated analytically and numerically. The general approach to prevent mentioned numerical problems is described. It is based on analytical fitting with additional constraints considering analytical properties of the hydrodynamic variables.

It was shown that the proposed approach allows significantly to improve numerical solution in the problem regions. But some problems are remaining. It is necessary to improve execution of the program to extend its time from about ≈10…15 ns until a few microseconds.

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