

ANALYTICAL EXPRESSION FOR ELECTRON TRAJECTORY IN ACTIVE PLASMA LENS

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An active plasma lens is the focusing device that is used in the wakefield accelerators. Focusing properties of an active plasma lens usually are investigated using numerical methods. Here the analytical expression for the electron trajectory is presented. It may be used for the analytical investigation and estimation of plasma lens focusing properties. It may simplify recovering of the magnetic field distribution in a plasma lens using experimental results. To obtain it the momentum equation for an electron moving in the magnetic field in a plasma lens is used. The equation for an electron trajectory was derived. The case when the analytical solution may be obtained is presented. It is expressed through the Jacobi elliptic functions.

PACS: 41.75.Jv, 52.38.Kd, 51.20.+d, 52.25.Fi

INTRODUCTION

A laser and plasma wakefield allow to create the strong electric fields on the order of 100 GV/m to accelerate the electrons and positrons and build the compact acceleration structures for the free electron lasers and linear colliders [1–6]. For different applications the beams with small emittance are required and the focusing devices to deliver the accelerated beams are needed [1–4, 6]. The different focusing device may be used for this goal: solenoids, quadrupole magnetic lenses, that was mentioned in [1, 2]. The focusing field of solenoid is weak for relativistic beams due to $1/\gamma^2$ of focusing field. The quadrupoles provide stronger field ($1/\gamma$). The permanent quadrupole magnets may be compact, but their gradient is limited by 100 T/m. It focuses only in one plane while in perpendicular plane it defocuses.

An active plasma lens (APL) does not have these problems. The detailed description of an APL and its experimental investigation are presented, for example, in [1–6]. The plasma lens is the thin cylindrical capillary filled by gas, usually hydrogen or argon. In the capillary the electric discharge is created and the electric current passes along capillary. This current is excited by external high voltage applied to the two plane electrodes located on the both capillary ends. The capillary radius may be different from a few hundreds μm up to 1...1.5 mm. The capillary length may be usually a few cm. The longitudinal current generates azimuthal focusing of magnetic field with gradient of order a few kT/m. The value of the discharge current is changed from hundreds of Amperes to a few thousands of Amperes. The advantages of an APL comparing to the quadrupoles are the small size and more significant magnetic field gradient. In addition, an APL has the axisymmetric structure of the focusing forces.

On another hand, unfortunately, an APL has also some problems. These are the aberrations that is caused by nonuniform distribution of the magnetic field gradient, that leads to the emittance growth. The reason of magnetic field gradient nonuniformity is the nonuniform distribution of plasma temperature in the capillary discharge, where the temperature decreases from the hot capillary center to the cold capillary wall. The steady-state conditions are formed in the plasma lens due to the balance between the Ohmic plasma heating

and thermal losses on the capillary wall. The corresponding physical model (called JT-model) was presented in [5]. In this paper the temperature radial distribution was obtained and the relationship between the temperature and discharge current density is presented. This dependence is $J(r) \sim T(r)^{3/2}$. In [1, 4] the plasma temperature distribution, using JT-model was investigated. The discharge current and azimuthal magnetic field distribution expression are presented. An expression for dependence of the magnetic field on the radius can be presented as

$$B(r) = \frac{\mu_0 I_0}{2\pi R} (-\alpha_1 r + \alpha_3 r^3 + \alpha_5 r^5 + \dots), \quad (1)$$

where $\frac{\mu_0 I_0}{2\pi R}$ is the magnetic field on the capillary wall, μ_0 is the vacuum permeability, I_0 is the total discharge current, R is the capillary radius, r is the dimensionless radius $r \rightarrow r/R$, $\alpha_1, \alpha_3, \alpha_5, \dots$ are the coefficients. It is necessary to note, that here an expression signs is opposite to one in [1,4]. The reason is the necessity to provide the right magnetic force direction to the capillary axis in the equation describing an electron movement in the plasma lens.

In [1, 2] the result of the experimental measurements of the magnetic field distribution in the plasma lens is presented. The mathematical algorithm of the experimental result processing is described in details in [2]. In the first approximation it is assumed that the magnetic field distribution gradient is uniform. In this case it is defined as a solution of the nonlinear algebraic equation. In another case, an assumption, when the gradient is not uniform the algorithm is more complex. The fitting method for processing of the experimental data is used. This approach requires large number of numerical calculations to find the best trajectories closed to the experimental results.

To improve this algorithm the analytical expression for the electron trajectory, containing the limit number of parameters, describing the magnetic field may be used. In this case the definition of these parameters is required instead the calculation of different possible trajectories to find the best of them.

In this report the analytical solution for the electron trajectory in the radial magnetic field of type (1) with the coefficients α_1, α_3 are considered.

1. PROBLEM

Numerical methods are used mainly for the focusing properties investigation of an APL. To define experimentally the magnetic field distribution in an APL in the case of the nonuniform gradient the numerical fitting to the experimental data requires a large number of the computational works (see [1–3]). In the modern electronic optics, the wide range of the numerical and analytical methods are used. They presented, for example, in [7]. An analytical solution for the electron trajectory in the focusing devices allows quickly to obtain qualitatively the information about the focusing properties. In most cases the trajectory passes close to the optics axis and has the small angle with it. It is the so-called paraxial approximation.

In an APL the azimuthal magnetic field, created by the longitudinal discharge current, causes the axially symmetric focusing force being directed to the capillary axis. Usually, the capillary radius is much lower than its length, so the paraxial approximation is suitable in this case.

We are interested to obtain the analytical expression for the electron trajectory in an APL. The ultra relativistic electron movement is considered. The magnetic field is assumed to be as independent from time. To obtain the needed expression for the electron trajectory, the momentum equation is used. The magnetic field distribution is given analytically by the formula that is similar to (1) where only the linear and cubic terms are taken into account. The odd powers of radius are used because magnetic field is odd function.

Further the momentum equation will be simplified and transformed to one equation for the electron trajectory. The new dimensionless variables will be introduced. The obtained trajectory equation may be reduced to the nonlinear Duffing pendulum equation with cubic nonlinearity. Its solution is presented by the Jacobi elliptic functions.

2. ANALYTICAL EXPRESSION FOR ELECTRON TRAJECTORY

We consider the electron movement in the stationary magnetic field in the thin capillary. As it was mentioned above the axially symmetric azimuthal magnetic field is created by the longitudinal discharge current passing along the capillary filled by plasmas. The corresponding momentum equation has the next form

$$\frac{d\mathbf{p}}{dt} = -e\mathbf{v} \times \mathbf{B}, \quad \mathbf{p} = m_e\gamma\mathbf{v}, \quad (2)$$

where \mathbf{p} is the electron momentum, \mathbf{v} is an electron velocity, \mathbf{B} is the magnetic field, m_e is an electron mass, $\gamma = \sqrt{1 - v^2/c^2}$ is the relativistic factor, c is the light velocity. We will consider the movement, that take place in plane passing through the capillary axis, thus the azimuthal velocity component is equal to zero. In this case the plane Cartesian coordinates may be used. Let us direct the z -axis along the capillary axis. Then the x -axis is directed perpendicular to the capillary axis. Thus, we have the following velocity components: v_z and v_x . The magnetic field contains the B_y component.

The vector equation (2) may be presented in the scalar form

$$\begin{cases} \frac{dv_x}{dt} = \frac{e}{m_e\gamma} v_z B_y, \\ \frac{dv_z}{dt} = -\frac{e}{m_e\gamma} v_x B_y. \end{cases} \quad (3)$$

Due to the electron movement in the magnetic field, its energy is conserved, consequently the relativistic factor is constant. From this condition the next relation is the following

$$v_x^2 + v_z^2 = c^2 \frac{\gamma^2 - 1}{\gamma^2}. \quad (4)$$

Let us make changing of the independent variable: the current longitudinal coordinate z instead of the time t , that is expressed by the next relation

$$\frac{d}{dt} = v_z \frac{d}{dz}. \quad (5)$$

Further the set of equations (3) becomes as

$$\begin{cases} \frac{dv_x}{dz} = \frac{e}{m_e\gamma} B_y, \\ \frac{dv_z}{dz} v_z = -\frac{e}{m_e\gamma} v_x B_y. \end{cases} \quad (6)$$

The expression (4) is the movement integral, it follows, in particular, from (6). Taking into account (4) and (5) the expression for v_x is obtained as

$$v_x = \frac{c}{\gamma} \frac{dx}{dz} \sqrt{\frac{\gamma^2}{1 + \left(\frac{dx}{dz}\right)^2}}. \quad (7)$$

Substituting (7) into the first equation (6), the next equation is obtained:

$$\frac{d^2x}{dz^2} = \frac{e}{m_e c \sqrt{\gamma^2 - 1}} B_y(x) \left[1 + \left(\frac{dx}{dz}\right)^2 \right]^{\frac{3}{2}}. \quad (8)$$

The expression (8) is the equation describing the electron trajectory in an APL.

The above equations (2)-(8) were suitable for the plane movement and Cartesian plane coordinate system that was used. The considered problem is the axially symmetric, so the coordinate x may be changed by r and instead $B_y(x)$ the designation $B(r)$ will be used. Let us introduce the dimensionless longitudinal coordinate and radius in the next way: $z \rightarrow \frac{z}{L}$, $r \rightarrow r/R$, where L is capillary length and R is capillary radius, as it was mentioned above. As it follows from (1) the radial magnetic field distribution may be presented as a power series on the odd radius degrees. Further, we will be limiting ourselves only to one nonlinear (cubic) term. The analytical expression for the magnetic field distribution has the next form

$$B(r) = \frac{\mu_0 I_0}{2\pi R} (-\alpha_1 r + \alpha_3 r^3). \quad (9)$$

The coefficient α_1 is on the order of one. The trajectory equation in dimensionless variables has the following form

$$\frac{d^2r}{dz^2} = A_0 (-\alpha_1 r + \alpha_3 r^3) \left[1 + \frac{r_0^2}{L^2} \left(\frac{dr}{dz}\right)^2 \right]^{\frac{3}{2}}, \quad (10)$$

where $A_0 = A_1 A_2 B_0$, $A_1 = \frac{e}{m_e c \sqrt{\gamma^2 - 1}}$, $A_2 = L^2/R$, $B_0 = \frac{\mu_0 I_0}{2\pi R}$.

Taking into account that $R \ll L$, the next condition is valid

$$\frac{R^2}{L^2} \left(\frac{dr}{dz}\right)^2 \ll 1. \quad (11)$$

Due to that, the last brackets in (10) may be omitted, and equation (10) becomes

$$\frac{d^2r}{dz^2} = A_0 (-\alpha_1 r + \alpha_3 r^3). \quad (12)$$

The equation (12) is the nonlinear oscillator equation. It should be supplemented by initial conditions

$$r(0) = r_0, \quad r'(0) = r'_0, \quad (13)$$

where r_0 is the electron radius on the APL entrance, and r'_0 , is the tangent of the angle of inclination to the capillary axis.

To solve the equation (12) it may be multiplied at the both sides by $\frac{dr}{dz}$ and integrated. As a result, we obtain the equation of the first order

$$\left(\frac{dr}{dz}\right)^2 = -A_0\alpha_1 r^2 + A_0\alpha_3 \frac{r^4}{2} + D, \quad (14)$$

where D is the integration constant that is equal-to

$$D = r'_0{}^2 + A_0\alpha_1 r_0^2 - A_0\alpha_3 \frac{r_0^4}{2}.$$

Further, we will consider some details of obtaining the analytical solution for the equation (12). Equation (14) may be transformed to the following form:

$$\int_{r_0}^r \frac{dx}{\sqrt{A_0\alpha_3 \frac{x^4}{2} - A_0\alpha_1 x^2 + D}} = z. \quad (15)$$

Last equation may be presented as following

$$\int_{r_0}^r \frac{dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \sqrt{\frac{A_0\alpha_3}{2}} z, \quad (16)$$

where

$$a^2, b^2 = \frac{\alpha_1}{\alpha_3} \pm \sqrt{\left(\frac{\alpha_1}{\alpha_3} - r_0^2\right)^2 - \frac{2}{A_0\alpha_3} r_0'^2} \quad (17)$$

are the roots of right part of Eq. (14). The solution for equations (12) and (14) with initial condition on the entrance of an APL is

$$r(z) = b \cdot \text{sn}\left(a\sqrt{\frac{A_0\alpha_3}{2}}z + \text{sn}^{-1}\left(\frac{r_0}{b}, \frac{b}{a}\right), \frac{b}{a}\right) \quad (18)$$

where $\text{sn}(z, k)$ is the Jacobian elliptic function of argument z , k is the module of the Jacobian elliptic function, $\text{sn}^{-1}(z, k)$ is the inverse Jacobian elliptic function. The Jacobian elliptic functions and their properties are presented in detail in [8].

If the coefficient $\alpha_3 = 0$, the equations (12) and (14) describe the linear oscillator and their solution for initial conditions (13) is

$$r(z) = r_0 \cos(\lambda z) + \frac{r'_0}{\lambda} \sin(\lambda z), \quad \lambda = \sqrt{A_0\alpha_1}. \quad (19)$$

If $k \ll 1$ the Jacobian elliptic functions approximately may be presented by the next expression

$$\text{sn}(z, k) \approx \sin(z) - 0.25k^2(z - \sin(z)\cos(z))\cos(z). \quad (20)$$

3. PRELIMINARY NUMERICAL RESULTS

The preliminary numerical analysis was carried out for the following parameters of an APL: the capillary radius is equal to 0.5 mm, the capillary length is equals to 15 mm, the discharge current is equals to 400 A, the energy of injected in an APL electrons is equals to 200 MeV. In this case the magnetic field strength on the capillary wall is equals to 0.16 T, the relativistic factor is equal 391, the ratio R^2/L^2 is equal 0.001. The coefficients are $\alpha_1 = 1.3$, $\alpha_3 = 0.05$. The initial conditions are $r_0 = 0.6$ and $r'_0 = 0.0$.

The magnetic field distribution in an APL (described by the formula (9)) with the temperature gradient and the nonlinear magnetic field gradient has the shape approximately shown in Fig. 1.

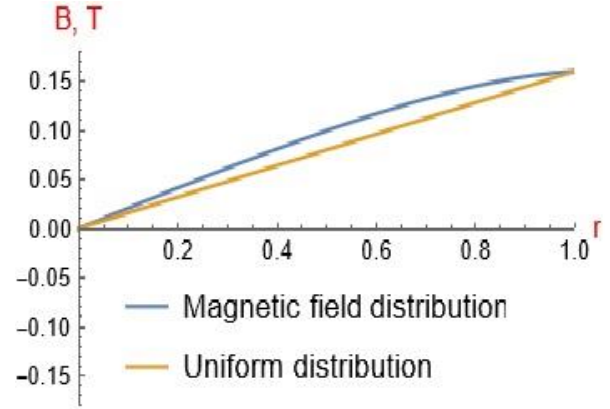


Fig. 1. Characteristic magnetic field distribution in an active plasma lens with the nonlinear gradient. The similar magnetic field distributions were obtained numerically and experimentally and were presented in [1, 3, 4]

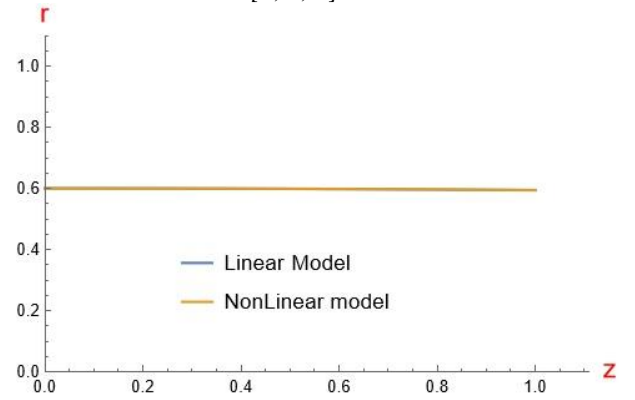


Fig. 2. Trajectories of an electron in an APL. Here presented as linear model ($\alpha_1 = 0.0$) as nonlinear ($\alpha_3 = 0.05$). First of them is described by formula (19), second by formula (18).

The both curves visually coincide

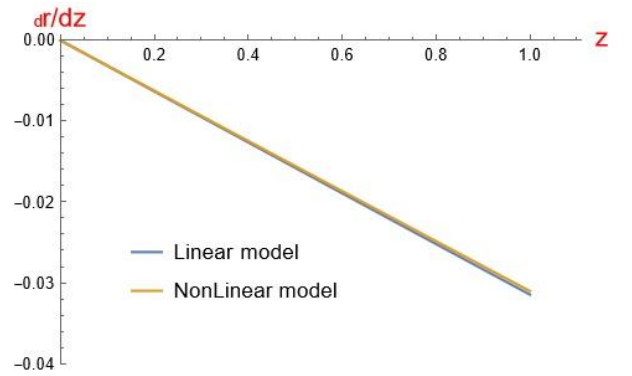


Fig. 3. Derivatives for the electron trajectories in an APL. Two cases present linear model ($\alpha_3 = 0.0$) and nonlinear one ($\alpha_3 = 0.05$). First of them is described by formula (19), second by formula (18)

The similar shapes of the magnetic field distribution are presented in [1] (see Fig. 1), [2] (see Fig. 2) and [4] (see Fig. 1,c), obtained numerically and experimentally. This means that the magnetic field distribution, described by formula (9) approximately may present magnetic field distribution in a real APL and the analytical solution (18) may be used for the investigation of focusing properties

of an APL. The numerical investigation of the formula (1) shows that taking into account the term with 5th power may better present the magnetic field distribution. As it follows from the analytical analysis in this case, the analytical solution may be expressed by the elliptic functions, but it is more complex mathematical problem.

The electron trajectories in Fig. 2 and their derivatives in Fig. 3 in an APL are presented. The linear (19) and nonlinear solutions (18) are presented. Initial conditions are the same. The trajectories are visually indistinguishable. Difference is in the 5th sign. On another hand there is visible the small difference in the derivatives in Fig. 3. For the derivatives difference is in 4th sign on the APL output.

Let us consider this results in detail. The module of Jacobian elliptic functions is equal to $a/b=0.083$. Thus, the second term in the expansion (20) of this function is very small. If to compare the entrance point coordinate $r_0 = 0.6$ of an electron with Fig. 1, it is seen that this point is located on almost the linear part of the blue curve for the magnetic field distribution. Here the nonlinearity influence is almost negligible.

As it is seen, to get satisfactory nonlinearity resolution in the experiment the enough distance between an APL and registering device is required. In principle, the corresponding estimations may be obtained from the presented expression. To define magnetic field distribution from the experimental data (18) and (20) may be used. It will be needed to calculate only coefficients α_1 and α_3 . The least square method may be used to define these coefficients to fit calculation result to the experimental data.

CONCLUSIONS

In this report an analytical expression for the electron trajectory in an active plasma lens with nonuniform

magnetic field is presented. Only the first nonlinear (cubic) term is used. The solution for the trajectory is expressed through the Jacobi elliptic functions. The brief analysis of the obtained solution is presented.

ACKNOWLEDGEMENTS

Author would like to thank for useful discussions M. Thevenet, V.I. Maslov, and I.A. Vartanians.

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